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On Convergence of Interpolation to Analytic Functions[1]

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In the present paper, both the perfect convergence for the Lagrange interpolation of analytic functions on $[-1, 1]$ and the perfect convergence for the trigonometric interpolation of analytic functions on $[-\pi, \pi]$ with period 2π are discussed.

1. INTRODUCTION

The convergence for the Lagrange interpolation of analytic functions on $[-1, 1]$ has become a topic of intense research. A sufficiently extensive literature on the subject is contained in [1, 2]. Until quite recently the research on convergence for the trigonometric interpolation of periodic analytic functions has been completely ignored, although it is well known that for the quadrature formulas of singular integrals with Hilbert kernel it can yield excellent results [3]. In the present paper, we first give the result on the perfect convergence of the Lagrange interpolation for analytic functions on $[-1, 1]$, which improves an important result in [1]. Then we discuss the perfect convergence for the trigonometric interpolation of analytic functions on $[-\pi, \pi]$ with period 2π. By using a good technique we reduce the trigonometric interpolation for some special analytic functions on $[-\pi, \pi]$ with period 2π to the Lagrange interpolation for some analytic functions on $[-1, 1]$ and the perfect convergence theorem for the trigonometric interpolation of analytic functions on $[-\pi, \pi]$ with period 2π follows from this.

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2. LAGRANGE INTERPOLATION

Let f be a function defined on $[-1, 1]$ and

$$\Pi_n\colon x_{n,1} < x_{n,2} < \cdots < x_{n,n} \qquad (-1 \leqslant x_{n,1}, x_{n,n} \leqslant 1) \tag{2.1}$$

be a set of n distinct points of the interval $[-1, 1]$. The monic polynomial of degree n with the zeros $x_{n,j}$ still is denoted by Π_n, i.e., $\Pi_n(x) = \prod_{j=1}^{n} (x - x_{n,j})$. It is well known that the Lagrange interpolating polynomial of f corresponding to the nodal set Π_n takes the form of

$$(L_n^{\Pi} f)(x) = \sum_{j=1}^{n} f(x_{n,j}) \frac{\Pi_n(x)}{\Pi_n'(x_{n,j})(x - x_{n,j})}. \tag{2.2}$$

Let $\{\Pi_n\}_1^\infty$ be a sequence of nodal sets and $\|\cdot\|$ denote the Chebyshev norm of a continuous function defined in $[-1, 1]$. If $\lim_{n\to\infty} \|f - L_n^{\Pi} f\| = 0$, then we say that the Lagrange interpolation of f corresponding to the sequence of nodal sets Π_n is convergent. We now discuss the convergence of $L_n^{\Pi} f$ for the analytic functions on $[-1, 1]$. In what follows, U always denotes a neighborhood of $[-1, 1]$ and Ω always denotes a closed set which contains $[-1, 1]$. If f is analytic on U, we then write $f \in A(U)$. If there is a neighborhood U of Ω such that $f \in A(U)$, then we write $f \in A(\Omega)$. In addition, we will frequently use the simple fact below.

LEMMA 2.1. *Let $B(x, 1+|x|) = \{z\colon |z-x| < 1+|x|\}$, then*

$$\mathscr{U}_0 = \bigcup_{x\in[-1,1]} B(x, 1+|x|) = B(-1, 2) \bigcup B(1, 2) = \bigcup_{x\in(-1,1)} B(x, 1+|x|). \tag{2.3}$$

Proof. For each $x \in [0, 1]$, if $z \in B(x, 1+|x|)$ then $|z-1| \leqslant |z-x| + |1-x| < 2$, that is $\bigcup_{x\in[0,1]} B(x, 1+|x|) \subset B(1, 2)$. Conversely, for each $z \in B(1, 2)$, there is some $x_0 \in [0, 1)$ such that $|z - x_0| < 1 + |x_0|$ since $\lim_{x\to 1^-} |z-x| - (1+|x|) < 0$, that is, $B(1, 2) \subset \bigcup_{x\in[0,1)} B(x, 1+|x|)$. Thus $\bigcup_{x\in[0,1]} B(x, 1+|x|) = B(1, 2) = \bigcup_{x\in[0,1)} B(x, 1+|x|)$. Similarly, $\bigcup_{x\in[-1,0]} B(x, 1+|x|) = B(-1, 2) = \bigcup_{x\in(-1,0]} B(x, 1+|x|)$.

THEOREM 2.1. *If f is analytic in the closure of $\mathscr{U}_0$ then $\lim_{n\to\infty} \|f - L_n^{\Pi} f\| = 0$ for all choices of nodal sets Π_n.*

Proof. Obviously there exists $\varepsilon > 0$ such that $f \in A(\overline{U_\varepsilon})$, where

$$U_\varepsilon = \bigcup_{x\in[-1,1]} B(x, 1+|x|+\varepsilon) = B(-1, 2+\varepsilon) \bigcup B(1, 2+\varepsilon)$$

is the open. We now have [1, 4]

$$f(x)-(L_n^{\Pi} f)(x)=\frac{1}{2\pi i}\int_{\partial U_\varepsilon}\frac{\Pi_n(x)}{\Pi_n(z)}\frac{f(z)}{z-x}\,dz, \qquad x\in[-1,1].$$

Notice that

$$\left|\frac{\Pi_n(x)}{\Pi_n(z)}\right|\leqslant\prod_{k=1}^{n}\frac{1+|x_{n,k}|}{1+|x_{n,k}|+\varepsilon}\leqslant\left(\frac{1}{1+\varepsilon/2}\right)^n, \qquad x\in[-1,1], \qquad z\in\partial U_\varepsilon,$$

then we get $\|f-L_n^{\Pi} f\|=O(1)(1+\varepsilon/2)^{-n}=o(1)$ as $n\to\infty$.

It must be noted that Theorem 2.1 is not always true for $f\in A[-1,1]$. References [1, 2] have shown that (Kalmàr–Walsh theorem) the Lagrange interpolation corresponding to the sequence of nodal sets Π_n is convergent for each $f\in A[-1,1]$ if and only if the sequence $\{\Pi_n\}_1^\infty$ is distributed according to the arcsine distribution. In other words, if the Lagrange interpolation of $f\in A[-1,1]$ is convergent for all choices of nodal sets Π_n, then $f|_{[-1,1]}$ must has an analytic continuation into a sufficiently large neighborhood of $[-1,1]$ where $f|_{[-1,1]}$ is the restricted function of f on $[-1,1]$. In fact, $\mathscr{U}_0$ is the smallest one, which will be stated later in Theorem 2.2 in detail.

LEMMA 2.2. *Let f be a function on $[-1,1]$ and analytic at the point $x_0\in[-1,1]$, then for each n there is $\delta_n>0$, such that $\|L_n^{\Pi} f-T_n f\|<\frac{1}{n}$ if all nodes $x_{n,j}$ in Π_n satisfy $|x_{n,j}-x_0|<\delta_n$, where $T_n f$ is the Taylor polynomial of degree $(n-1)$ of f at the point x_0. Such scheme of nodal sets Π_n is called of Taylor type at x_0.*

Remark 2.1. We say that f is analytic at x_0 if there is a neighborhood $U(x_0)$ of x_0 such that $f|_{[-1,1]\cap U(x_0)}$ has an analytic continuation into $U(x_0)$.

Proof. We know that [5], Newton's form of the operator L_n^{Π} is

$$(L_n^{\Pi} f)(x)=f(x_{n,1})+\sum_{j=1}^{n-1} f[x_{n,1}, x_{n,2}, \ldots, x_{n,j+1}]\prod_{k=1}^{j}(x-x_{n,k})$$

and the jth order divided differences satisfies

$$f[x_{n,1}, x_{n,2}, \ldots, x_{n,j+1}]=\frac{1}{j!}f^{(j)}(\xi_{n,j}), \qquad \exists\xi_{n,j}\in[x_{n,1}, x_{n,j+1}],$$

$$j=1, 2, \ldots, n-1.$$

From the above relations, the assertion is clear.

LEMMA 2.3. *Suppose that f is defined on $[-1, 1]$ and analytic at the point x_0 where $x_0 \in [-1, 1]$. If there is a scheme of nodal sets Π_n such that both $\lim_{n\to\infty} \|f - L_n^{\Pi} f\| = 0$ and $\lim_{n\to\infty} \|L_n^{\Pi} f - T_n f\| = 0$ where $T_n f$ is the Taylor polynomial of degree $(n-1)$ of f at x_0, then $f|_{[-1,1)}$ has an analytic continuation into $B(x_0, 1+|x_0|)$ for $x_0 \in [-1, 0)$, $f|_{(-1,1]}$ has an analytic continuation into $B(x_0, 1+|x_0|)$ for $x_0 \in (0, 1]$ and $f|_{(-1,1)}$ has an analytic continuation into $B(0, 1)$ for $x_0 = 0$.*

Proof. Clearly, $\lim_{n\to\infty} \|f - T_n f\| = 0$. That is to say that the Taylor series of f at x_0 also converges to $f(x)$ at each point $x \in [-1, 1]$, hence it is an analytic function on $B(x_0, 1+|x_0|)$. In other words, $f|_{(-1,1]}$ has just an analytic continuation into $B(x_0, 1+|x_0|)$ for $x_0 \in (0, 1]$ since $(-1, 1] \subset B(x_0, 1+|x_0|)$ in this case. The other cases are treated similarly.

Remark 2.2. We point out an important special case which will be used in the next section. Let f be an even function on $[-1, 1]$ and analytic at the points 0. If there is a scheme of nodal sets Π_{2m-1} such that both $\lim_{m\to\infty} |f(x) - (L_{2m-1}^{\Pi} f)(x)| = 0$ for each $x \in (-1, 1)$ and $\lim_{m\to\infty} \|L_{2m-1}^{\Pi} f - T_{2m-1} f\| = 0$ where $T_n f$ is the Taylor polynomial of degree $(n-1)$ of f at the point 0, then $f|_{(-1,1)}$ has an analytic continuation into $B(0, 1)$. In fact, we have $\lim_{m\to\infty} |f(x) - (T_{2m-1} f)(x)| = 0$ for each $x \in (-1, 1)$, hence $\lim_{n\to\infty} |f(x) - (T_n f)(x)| = 0$ for each $x \in (-1, 1)$ since f is an even function. In exactly the same way as in the proof of Lemma 2.3, the conclusion is now obtained.

THEOREM 2.2. *Suppose that f is defined on $[-1, 1]$, analytic at -1 and 1. If $\lim_{n\to\infty} \|f - L_n^{\Pi} f\| = 0$ for all choices of nodal sets Π_n, then $f|_{[-1,1]}$ has an analytic continuation into $\mathscr{U}_0$.*

Proof. By Lemma 2.2 and Lemma 2.3, we know that $f|_{(-1,1]}$ has an analytic continuation into $B(1, 2)$ and $f|_{[-1,1)}$ has an analytic continuation into $B(-1, 2)$. This is to say that $f|_{[-1,1]}$ has an analytic continuation into $\mathscr{U}_0$.

In the same manner, and noticing Lemma 2.1 we also have

THEOREM 2.3. *Suppose that f is defined on $[-1, 1]$ and analytic in $(-1, 1)$. If $\lim_{n\to\infty} \|f - L_n^{\Pi} f\| = 0$ for all choices of nodal sets Π_n, then $f|_{[-1,1]}$ has an analytic continuation into $\mathscr{U}_0$.*

THEOREM 2.4. $\lim_{n\to\infty} \|f - L_n^{\Pi} f\| = 0$ *for each $f \in A(\Omega)$ and all choices of nodal sets Π_n, in which case it is said that the Lagrange interpolation is perfectly convergent for $A(\Omega)$, if and only if $\mathscr{U}_0 \subset \Omega$.*

Proof. Here we only need to prove "only if." If $\mathscr{U}_0 \not\subset \Omega$, there exists some $z_0 \in \mathscr{U}_0/\Omega$, so $f(z) = 1/(z-z_0) \in A(\Omega) \subseteq A[-1, 1]$. The proof follows from the preceding Theorem 2.2.

3. TRIGONOMETRIC INTERPOLATION

Let

$$\Delta_n\colon t_{n,1} < t_{n,2} < \cdots < t_{n,n} \qquad (-\pi \leqslant t_{n,1},\ t_{n,n} < \pi) \tag{3.1}$$

be a set of n distinct points of the interval $[-\pi, \pi)$, we still use the Δ_n to denote the following (semi) trigonometric polynomial

$$\Delta_n(t) = \prod_{j=1}^{n} \sin \frac{t-t_{n,j}}{2}. \tag{3.2}$$

For functions f defined on $[-\pi, \pi]$, we introduce the trigonometric interpolation operator (TIO) of f corresponding to the nodal set Δ_n as

$$(T_n^{\Delta} f)(t) = \sum_{j=1}^{n} T_{n,j}^{\Delta}(t)\, f(t_{n,j}), \tag{3.3}$$

where

$$T_{n,j}^{\Delta}(t) = \begin{cases} \dfrac{\Delta_n(t)}{2\Delta_n'(t_{n,j})} \csc \dfrac{t-t_{n,j}}{2}, & \text{if } n \text{ is odd}, \\[2ex] \dfrac{\Delta_n(t)}{2\Delta_n'(t_{n,j})} \cot \dfrac{t-t_{n,j}}{2}, & \text{if } n \text{ is even}. \end{cases} \tag{3.4}$$

Let $\|\cdot\|_r$ denote the Chebyshev norm of a 2π–periodic continuous function on the line segment $z = t + ir(-\pi \leqslant t \leqslant \pi, r \text{ real})$, in particular, write $\|\cdot\| = \|\cdot\|_0$. If the function f is a 2π-periodic analytic function on the closed rectangular set $D_r = \{z\colon |\operatorname{Re} z| \leqslant \pi, |\operatorname{Im} z| \leqslant r\}(r > 0)$ then we write $f \in AP(D_r)$. In addition, write $AP[-\pi, \pi] = \bigcup_{r>0} AP(D_r)$. Let

$$R_0 = 2\ln(1+\sqrt{2}) = 2 \operatorname{arcsinh} 1. \tag{3.5}$$

Du obtained the following [3]

THEOREM 3.1. *If $f \in AP(D_r)$ and $r \geqslant R_0$ then $\lim_{n\to\infty} \|f - T_n^{\Delta} f\| = 0$ for all choices of nodal sets Δ_n.*

In [6], Liu and Du obtained the following theorem.

THEOREM 3.2. *$\lim_{n\to\infty} \|f - T_n^{\Delta} f\| = 0$ for each $f \in AP[-\pi, \pi]$, iff the sequence of the nodal sets Δ_n is uniformly distributed, i.e., $\lim_{n\to\infty} \sqrt[n]{\|\Delta_n\|} = \frac{1}{2}$.*

We very easily find some sequence of nodal sets Δ_n which is not uniformly distributed [6]. This fact tells us that if $\lim_{n\to\infty} \|f - T_n^{\Delta} f\| = 0$ for each $f \in AP(D_r)$ and all choices of nodal sets Δ_n, in which case it is said that the trigonometric interpolation to be perfectly convergent for $AP(D_r)$, then r must be also restricted by some condition. In fact, the converse theorem of Theorem 3.1 still holds.

THEOREM 3.3. *The trigonometric interpolation is perfectly convergent for $AP(D_r)$ if and only if $r \geqslant R_0$.*

We first establish some lemmas. To do so, if $f(t)$ is a function defined on $[-\pi, \pi]$, then let

$$f^*(x) = f(2 \arcsin x),\ x \in [-1, 1], \tag{3.6}$$

which is called the associated function of f. A nodal set $\Pi_n = \{x_{n,1}, x_{n,2}, ..., x_{n,n}\}$ on $(-1, 1)$ is said to be normally symmetric if its nodes is symmetric about 0, i.e., $x_{n,j} = -x_{n,n+1-j}$ $(j = 1, 2, ..., n)$. Similarly, the meaning of a normally symmetrical nodal set $\Delta_n = \{t_{n,1}, t_{n,2}, ..., t_{n,n}\}$ on $(-\pi, \pi)$ is obvious.

LEMMA 3.1. *f^* is an even function on $[-1, 1]$ if and only if f is an even function on $[-\pi, \pi]$. The nodal set $\Pi_n = \{x_{n,1}, ..., x_{n,n}\}$ on $(-1, 1)$ is normally symmetrical if and only if the nodal set $\Delta_n = \{2 \arcsin(x_{n,1}/2), ..., 2 \arcsin(x_{n,n}/2)\}$ is normally symmetrical. In addition, $\|f - T_n^{\Delta} f\|_0 = \|f^* - L_n^{\Pi} f^*\|$ if f^* (f) is an even function and Π_n (Δ_n) is normally symmetrical.*

Proof. We only prove the later statement. $(L_n^{\Pi} f^*)(-x)$ is also the Lagrange interpolating polynomial of f corresponding to the nodal set Π_n since f^* is even and Π_n is normally symmetrical, hence $(L_n^{\Pi} f^*)(-x) = (L_n^{\Pi} f^*)(x)$. This is to say that $L_n^{\Pi} f^*$ is an even function. Thus, in either case $n = 2m$ or $n = 2m - 1$ we have $(L_n^{\Pi} f^*)(x) = \sum_{j=0}^{m-1} a_j x^{2j}$. So, $G(t) = (L_n^{\Pi} f^*)(\sin \frac{t}{2})$ is a trigonometric polynomial of degree at most $(m-1)$, and $G(t_{n,j}) = f^*(x_{n,j}) = f(t_{n,j})$, therefore [3], $(T_n^{\Delta} f)(t) = (L_n^{\Pi} f^*)(\sin \frac{t}{2})$. Moreover by $\sin(t/2)$ being $1-1$ from $[-\pi, \pi]$ to $[-1, 1]$, we know $\|f - T_n^{\Delta} f\|_0 = \|f^* - L_n^{\Pi} f^*\|$.

LEMMA 3.2. *f^* is analytic at the point $x_0 \in (-1, 1)$ if and only if f is analytic at the point $t_0 \in (-\pi, \pi)$ where $x_0 = \sin(t_0/2)$.*

Proof. Let

$$w = \phi(z) = \sin\frac{z}{2}. \tag{3.7}$$

It is easy to check that it maps, by $1-1$, the domain $\{z\colon -\pi < \operatorname{Re} z < \pi\}$ to the domain $\mathscr{C}/((-\infty, -1] \cup [1, +\infty))$ where $\mathscr{C}$ denotes the whole complex plane. So ϕ is biholomorphic [4]. Noticing $\phi^{-1}(w) = 2\arcsin w$ for $w \in (-1, 1)$, thus X_0 is a sufficiently small neighborhood of x_0 iff T_0 is a sufficiently small neighborhood of t_0 where $X_0 = \phi(T_0)$. Moreover, if f is analytic at t_0, namely $f|_{[-\pi,\pi]\cap T_0}$ has an analytic continuation F into T_0, clearly $F \circ \phi^{-1}$ is just an analytic continuation of $f^*|_{[-1,1]\cap X_0}$ into X_0, this is to say that f^* is analytic at x_0, and vice versa.

LEMMA 3.3. *Let f be an even function on $[-\pi, \pi]$ and analytic at the point* 0. *If* $\lim_{n\to\infty} \|f - T_n^{\Delta} f\| = 0$ *for each scheme of normally symmetrical nodal sets Δ_n, then $f|_{(-\pi,\pi)}$ can be analytically extended into* $\mathscr{S}(0) = \{z\colon |\sin(z/2)| < 1\}$.

Proof. By Lemma 3.1 and Lemma 3.2 we know that f^* is an even function and analytic at the point 0. By using Lemma 2.2 we can construct a scheme of nodal sets Π_n such that it is of Taylor type and each Π_n is normally symmetrical. By using Lemma 3.1 and Lemma 2.3, $f^*|_{(-1,1)}$ has an analytic continuation into $B(0, 1)$. Noticing that ϕ in (3.7) is biholomorphic, we know that $f|_{(-\pi,\pi)}$ can be analytically extended into $\phi^{-1}(B(0, 1)) = \mathscr{S}(0)$.

Proof of Theorem 3.3. We only need to prove "only if." If $r < R_0$, we take $r < r_0 < R_0$, $z_0 = ir_0$. So $f(z) = (\sin^2\frac{z}{2} - \sin^2\frac{z_0}{2})^{-1} \in AP(D_r)$, and clearly $z_0 \in \mathscr{S}(0)$. Now the proof is obvious by Lemma 3.3.

We will carry over the result of Lemma 3.3 to the case of odd functions, which is not too easy.

LEMMA 3.4. *Let f be an odd function on $[-\pi, \pi]$ and analytic at the point* 0. *If* $\lim_{n\to\infty} \|f - T_n^{\Delta} f\| = 0$ *for each scheme of normally symmetric nodal sets Δ_n, then $f|_{(-\pi,\pi)}$ can be analytically extended into $\mathscr{S}(0)$.*

Proof. Step I. Let $F(t) = (\sin t) f(t)$. Then it is an even function on $[-\pi, \pi]$ and analytic at 0, its associated function F^* is also an even function on $[-1, 1]$ and analytic at 0 by Lemma 3.1 and Lemma 3.2. Let $\vartheta(x) = F^*(x)/(x^2 - 1)$ if $-1 < x < 1$ and $\vartheta(\pm 1) = 0$ (we may assign an arbitrary value). By using Lemma 2.2, we may construct a scheme of nodal sets Π_{2m+1} such that it is normally symmetrical and of Taylor type for ϑ at 0. Thus

$$\lim_{m\to\infty} \|T_{2m+1}\vartheta - L_{2m+1}^{\Pi}\vartheta\| = 0, \tag{3.8}$$

where $T_n \vartheta$ is the Taylor polynomial of degree $(n-1)$ of ϑ at 0. We now write those nodal sets by $\Pi_{2m+1} = \{x_{2m,1}, ..., x_{2m,m}, 0, x_{2m,m+1}, ..., x_{2m,2m}\}$.

Step II. Let $\Delta_{2m} = \{t_{2m,1}, t_{2m,2}, ..., t_{2m,2m}\}$ where $t_{2m,j} = 2 \arcsin x_{2m,j}$ with $x_{2m,j}$ as those above. Clearly, it is normally symmetrical since Π_{2m+1} is normally symmetrical. Let $\Delta^{\star}_{2(m+1)} = \{-\pi, t_{2m,1}, ..., t_{2m,m}, 0, t_{2m,m+1}, ..., t_{2m,2m}\}$. We show that $(T^{\Delta^{\star}}_{2(m+1)} F)(t) = \sin t (T^{\Delta}_{2m} f)(t)$ and it is an even trigonometrical polynomial of degree at most $(m+1)$. In fact, $(T^{\Delta^{\star}}_{2(m+1)} F)(t)$, $(T^{\Delta^{\star}}_{2(m+1)} F)(-t)$ and $(\sin t)(T^{\Delta}_{2m} f)(t)$ are all the trigonometric interpolating polynomial of F corresponding to the nodal set $\Delta^{\star}_{2(m+1)}$ in $H^T_{m+1}(\pi/2)$ [3, 7]. By Lemma 2.1 in [8] we have

$$(T^{\Delta^{\star}}_{2(m+1)} F)(t) = (T^{\Delta^{\star}}_{2(m+1)} F)(-t) = (\sin t)(T^{\Delta}_{2m} f)(t). \tag{3.9}$$

Thus

$$\|F - T^{\Delta^{\star}}_{2(m+1)} F\| \leqslant \|f - T^{\Delta}_{2m} f\| \to 0, \qquad \text{as} \quad m \to \infty. \tag{3.10}$$

For short, writing $G = T^{\Delta^{\star}}_{2(m+1)} F$, then, by (3.9) $G(t) = \sum_{j=0}^{m+1} b_j \cos jt$. We show easily that, for even trigonometric polynomials by the induction for their degree m, there is an even algebraic polynomial $H_{2(m+1)}$ of degree at most $2(m+1)$ such that $H_{2(m+1)}(\sin \frac{t}{2}) = G(t)$. Hence $H_{2(m+1)}$ is no other than the associated function G^* of G. Since $z = \sin(t/2)$ is $1-1$ from $[-\pi, \pi]$ to $[-1, 1]$ and by (3.10), we get

$$\lim_{m \to \infty} \|F^* - H_{2(m+1)}\| = 0. \tag{3.11}$$

Step III. Let $h_{2m}(x) = H_{2(m+1)}(x)/(x^2 - 1)$. Then it is an even algebraic polynomial of degree at most $2m$ since $H_{2(m+1)}(\pm 1) = G(\pm \pi) = F(-\pi) = 0$, and $h_{2m}(x) = \vartheta(x)$ if $x \in \Pi_{2m+1}$. Thus, we know $h_{2m} = L^{\Pi}_{2m+1} \vartheta$. From (3.11) we have

$$\lim_{m \to \infty} |\vartheta(x) - (L^{\Pi}_{2m+1} \vartheta)(x)| = 0, \qquad x \in (-1, 1). \tag{3.12}$$

Quoting Remark 2.2, by (3.8) and (3.12) we know that $\vartheta|_{(-1,1)}$ has an analytic continuation into $B(0, 1)$. Thus $F^*|_{(-1,1)}$ has an analytic continuation into $B(0, 1)$, by Lemma 3.2 $F|_{(-\pi,\pi)}$ has an analytic continuation into $\mathscr{S}(0)$, finally $f|_{(-\pi,\pi)}$ has an analytic continuation into $\mathscr{S}(0)$.

LEMMA 3.5. *Let f be a 2π–periodic function and analytic at the point t_0 where $t_0 \in (-\pi, \pi)$. If $\lim_{n \to \infty} \|f - T^{\Delta}_n f\| = 0$ for all choices of nodal sets Δ_n, then $f|_{(-\pi,\pi)}$ can be analytically extended into $\mathscr{S}(t_0) = \{z: |\sin \frac{1}{2}(z - t_0)| < 1\}$.*

Proof. *Case* I. If $t_0=0$, let $f^-(t)=f(-t)$, $f_e=\frac{1}{2}[f+f^-]$ and $f_o=\frac{1}{2}[f-f^-]$. Let Δ_n is normally symmetrical. Then $(T_n^\Delta f^-)(x)=(T_n^\Delta f)(-x)$ since they are both the trigonometric polynomial of f^- in $H_n^T(0)$ [3, 8]. Since $t\mapsto -t$ is $1-1$ from $[-\pi,\pi]$ to $[-\pi,\pi]$, so $\|f-T_n^\Delta f\|=\|f^- -T_n^\Delta f^-\|$, hence $\|f_e-T_n^\Delta f_e\|\leqslant\|f-T_n^\Delta f\|$ and $\|f_o-T_n^\Delta f_o\|\leqslant\|f-T_n^\Delta f\|$. Quoting Lemma 3.3 and Lemma 3.4, we know that, both f_e and f_o, so f can be analytically extended into $\mathscr{S}(0)$.

Case II. For general $x_0\in(-\pi,\pi)$, let $[t]_{2\pi}$ denote the number congruent to $t \pmod{2\pi}$ in $[-\pi,\pi)$ and let the mapping τ be defined by $t\mapsto[x_0+t]_{2\pi}$, clearly it is $1-1$ from $[-\pi,\pi)$ to $[-\pi,\pi)$. Let $F=f\circ\tau$ and $\Delta_n=\{t_{n,1},t_{n,2},\ldots,t_{n,n}\}\subset[-\pi,\pi)$ be a nodal set. Then $\Delta_n^\star=\{t_{n,1}^\star,t_{n,2}^\star,\ldots,t_{n,n}^\star\}$ is also a nodal set where $t_{n,j}^\star=\tau(t_{n,j})$ $(j=1,2,\ldots,n)$ and

$$\|f-T_n^{\Delta^\star}f\|=\|f\circ\tau-(T_n^{\Delta^\star}f)\circ\tau\|=\|F-T_n^\Delta F\|.$$

Now the proof is obviously reduced to Case I.

From the preceding Lemma 3.5, we obtain immediately the following

THEOREM 3.4. *If $f\in AP[-\pi,\pi]$ and $\lim_{n\to\infty}\|f-T_n^\Delta f\|=0$ for all choices of nodal sets Δ_n then $f|_{(-\pi,\pi)}$ has an analytic continuation into $\mathscr{D}_0=\{z\colon |\mathrm{Re}\, z|<\pi, |\mathrm{Im}\, z|<R_0\}$ with R_0 given in* (3.5).

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